SYSTEC RESEARCH CENTER FOR SYSTEMS & TECHNOLOGIES U.PORTO

# Reach and Robust Positively Invariant Sets Revisited

S. V. Rakovic (sasa.v.rakovic@tamu.edu),
F. A. C. C. Fontes (faf@fe.up.pt), and
I. V. Kolmanovsky (ilya@umich.edu)

*Abstract:* This paper considers linear dynamical systems subject to additive and bounded disturbances, and studies properties of their forward reach, robust positively invariant (RPI) and the minimal RPI sets. The analysis is carried out for discrete-time (DT), continuous-time (CT), and sampled-data (SD) systems from a unified perspective. In the DT and CT cases, we review key existing results, while for the SD case novel results that reveal substantial structural differences to the DT and CT cases are presented. In particular, the main topological and computational properties associated with the DT and CT forward reach and RPI sets fail to be directly applicable to SD systems. In light of this, we introduce and develop topologically compatible notions for the SD forward reach, RPI and mRPI sets. We address and enhance computational aspects associated with these sets by complementing them with approximate, but guaranteed, and numerically more plausible notions.

*Keywords*: Forward Reachability, Forward Reach Sets, Robust Positive Invariance, Robust Positively Invariant Sets, Minimal Robust Positively Invariant Sets, Bounded Disturbances, Discrete-Time, Continuous-Time, Sampled-data.

SYSTEC Report 2017-SC3, July 2017.

This is a preprint of an article that was mentioned as an output of research produced within the FCT project PTDC-EEI-AUT/2933-2014|16858—TOCCATA, namely in the activities report of Dr. Sasa Rakovic when he was a PostDoc scholarship holder at Univ. Porto, funded by the FCT project. This preprint is placed in the U.Porto Repository with open access to comply with the requirements of making the project research available.

# Reach and Robust Positively Invariant Sets Revisited

S. V. Raković [a,1], F. A. C. C. Fontes [b], and I. V. Kolmanovsky [c]

[a] *Texas A&M University, College Station, U.S.*

[b] *Universidade do Porto, Porto, Portugal*

[c] *The University of Michigan at Ann Arbor, Ann Arbor, U.S.*

**Abstract**

This paper considers linear dynamical systems subject to additive and bounded disturbances, and studies properties of their forward reach, robust positively invariant (RPI) and the minimal RPI sets. The analysis is carried out for discrete–time (DT), continuous–time (CT), and sampled–data (SD) systems from a unified perspective. In the DT and CT cases, we review key existing results, while for the SD case novel results that reveal substantial structural differences to the DT and CT cases are presented. In particular, the main topological and computational properties associated with the DT and CT forward reach and RPI sets fail to be directly applicable to SD systems. In light of this, we introduce and develop topologically compatible notions for the SD forward reach, RPI and mRPI sets. We address and enhance computational aspects associated with these sets by complementing them with approximate, but guaranteed, and numerically more plausible notions.

*Key words:* Forward Reachability, Forward Reach Sets, Robust Positive Invariance, Robust Positively Invariant Sets, Minimal Robust Positively Invariant Sets, Bounded Disturbances, Discrete–Time, Continuous–Time, Sampled–data.

## 1 Introduction

To enable deployment of increasingly complex and autonomous engineered systems, there is a rapidly growing interest in techniques which can assure that the operation of the system is confined to a known safe region. In particular, the study of reachability and invariance for systems subject to disturbances has been receiving much recent attention from the control community. Both from the perspective of contemporary control theory and from the perspective of practical applications, the significance of reachability and invariance analyses has been well-understood. Indeed, reachability and invariance are intimately linked with optimal control, set–membership state estimation, safety verification, and control synthesis under uncertainty. It has become a well–established fact that the analysis of uncertain constrained dynamics utilizing reachability and invariance enables one to guarantee *a–priori* relevant robustness properties such as robust constraint satisfaction, robust stability and convergence and recursive robust feasibility. An overview of these important research areas and main research topics can be found in [1–7], see also references therein. The previous studies on this subject have considered both DT and CT models, e.g., see the books [4, 5], the papers [8–10] for the treatment of DT case, and [11] for the treatment of CT case. Properties of the backward and forward reach sets, such as monotonicity, compactness, convexity, limiting behavior, have been instrumental to characterize and to compute related RPI sets, in general, and the maximal and minimal RPI sets, in particular. Some of the key results for DT and CT systems, in particular, the ones more relevant to the minimal RPI sets, are reviewed here from a unified perspective along with a novel framework for the analysis of forward reachability and robust positive invariance for sampled–data (SD) systems.

Within the setting of SD systems, the control is updated at discrete sampling instants while the evolution between sampling instances is modeled by an ordinary differential equation. This type of models is natural in many applications, especially when the variables of the plant evolve continuously in time yet the state measurements are updated at discrete–time instants and the controls are implemented, at the discrete–time instances, by a digital microcontroller. The SD setting is particularly important for constrained control problems, where the avoidance of inter–sample safety constraint violations should be guaranteed. When these problems are solved via discrete–time techniques, the state trajectory

[1] Corresponding author. E-mail address: sasa.v.rakovic@tamu.edu. Tel.: +1 512 998 7693.

might "jump over" obstacles between two discrete–time instants.

When considering SD systems, questions that arise are whether some properties established for DT and CT systems are still valid in the SD setting; or whether studying SD systems using approximate DT or CT models would allow us to safely conclude forward reachability and robust positive invariance properties, in particular within inter–sample intervals. We show here that, to a large extent, the answers to the previous questions are negative, reinforcing the relevance of developing specific methodologies within the setting of SD systems. In particular, the semi–group property as well as the preservation of positive invariance and anti–invariance that are valid in both the DT and CT cases, fail in the SD case. As a consequence, the main instrument to characterize and also compute the minimal RPI set in the DT and CT cases, i.e., the result saying that this set can be obtained as the unique fixed–point of a certain set dynamic equation [10], is no longer valid in the SD case. The analysis also shows that in the SD case, not only we loose the main instrument to compute the minimal RPI set, but also that such a set may not even exist. For these reasons, new notions of RPI and mRPI families of sets are first developed and then rendered computationally more attractive by introducing notions of RPI and mRPI pairs of sets.

**Paper Structure:** Sections 2 and 3 collect key results related to forward reachability and robust positive invariance in the DT and CT settings, respectively. Section 4 addresses the SD setting and follows deliberately a structure similar to the one of Sections 2 and 3 so as to highlight similarities and differences between the related results. A concluding discussion is delivered in Section 5.

**Nomenclature and Definitions:** The sets of nonnegative integers and real numbers are denoted by $\mathbb{Z}_{\geq 0}$ and $\mathbb{R}_{\geq 0}$, respectively. Any given sampling period $T \in \mathbb{R}_{\geq 0},\ T > 0$ induces sequences of sampling instances $\pi$ and sampling intervals $\theta$ both w.r.t. $\mathbb{R}_{\geq 0}$ specified via:

$$\pi := \{t_k\}_{k\in\mathbb{Z}_{\geq 0}} \text{ and } \theta := \{\mathcal{T}_k\}_{k\in\mathbb{Z}_{\geq 0}}, \text{ where } \forall k \in \mathbb{Z}_{\geq 0},\ t_{k+1} := t_k + T \text{ with } t_0 := 0 \text{ and } \mathcal{T}_k := [t_k, t_{k+1}).$$

The spectral radius and spectrum of a matrix $M \in \mathbb{R}^{n\times n}$ are denoted by $\rho(M)$ and $\sigma(M)$, respectively. The Minkowski sum of sets $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^n$ is given by

$$\mathcal{X} \oplus \mathcal{Y} := \{x + y\ :\ x \in \mathcal{X},\ y \in \mathcal{Y}\}.$$

Given a set $\mathcal{X}$ and a real matrix $M$ of compatible dimensions the image of $\mathcal{X}$ under $M$ is denoted by

$$M\mathcal{X} := \{Mx\ :\ x \in \mathcal{X}\}.$$

A set $\mathcal{X} \subset \mathbb{R}^n$ is *a $C$–set* if it is compact, convex, and contains the origin. A set $\mathcal{X} \subset \mathbb{R}^n$ is *a proper $C$–set* if it is a $C$–set and contains the origin in its interior.

Given any two compact sets $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^n$, their Hausdorff distance is defined by

$$\mathrm{H}_{\mathcal{L}}(\mathcal{X}, \mathcal{Y}) := \min_{\alpha\geq 0}\{\alpha\ :\ \mathcal{X} \subseteq \mathcal{Y} \oplus \alpha\mathcal{L} \text{ and } \mathcal{Y} \subseteq \mathcal{X} \oplus \alpha\mathcal{L}\},$$

where $\mathcal{L}$ is a given symmetric proper $C$–set in $\mathbb{R}^n$ inducing vector norm $|x|_{\mathcal{L}} := \min_\eta\{\eta\ :\ x \in \eta\mathcal{L},\ \eta \geq 0\}$.

For every $\tau$ in the real interval $[0, t]$, let $\mathcal{X}(\tau)$ be a subset of $\mathbb{R}^n$. The (set–valued) integral, a.k.a. the Aumann integral [12–14], $\int_0^t \mathcal{X}(\tau)d\tau$ of the set–valued function $\mathcal{X}(\cdot)$ over the interval $[0, t]$ is defined by:

$$\int_0^t \mathcal{X}(\tau)d\tau := \left\{\int_0^t x(\tau)d\tau : x(\cdot) \text{ is an IS of } \mathcal{X}(\cdot)\right\}.$$

Here, an IS (integrable selector) means that $x(\cdot)$ is $d\tau$–integrable and that $x(\tau) \in \mathcal{X}(\tau)$ $d\tau$–almost everywhere in the interval $[0, t]$. When we refer to "all" or "each" $\tau$ in $[0, t]$ we will mean "almost all".

Throughout this paper we work with nonempty sets, fixed sampling time $T \in \mathbb{R}_{\geq 0},\ T > 0$ and a fixed sequence of sampling instances $\pi$, unless stated otherwise.

## 2 The DT forward reach and minimal RPI sets

Consider linear dynamics described, for all $k \in \mathbb{Z}_{\geq 0}$, by:

$$x(t_{k+1}) = A_D x(t_k) + E_D w(t_k) \text{ with } w(t_k) \in \mathcal{W}_D, \tag{2.1}$$

where, for any $k \in \mathbb{Z}_{\geq 0}$, $x(t_k) \in \mathbb{R}^n$ and $w(t_k) \in \mathbb{R}^p$ are, respectively, the state and disturbance at time instance $t_k$, while the matrices $A_D \in \mathbb{R}^{n\times n}$ and $E_D \in \mathbb{R}^{n\times p}$ and the set $\mathcal{W}_D \subset \mathbb{R}^p$ are known exactly.

**Assumption 1** *The matrix $A_D$ is strictly stable (i.e., $\rho(A_D) < 1$). The matrix pair $(A_D, E_D)$ is controllable. The set $\mathcal{W}_D$ is a proper $C$–set in $\mathbb{R}^p$.*

The solutions to (2.1) satisfy, for all $k \in \mathbb{Z}_{\geq 0},\ k > 0$,

$$x(t_k) = A_D^k x + \sum_{i=0}^{k-1} A_D^{k-1-i} E_D w(t_i) \text{ with } x(t_0) = x. \tag{2.2}$$

The notions of the RPI and minimal RPI sets are recalled next for the sake of completeness.

**Remark 1** *A subset $\mathcal{S}$ of $\mathbb{R}^n$ is an RPI set for uncertain linear dynamics* (2.1) *if and only if for all $x \in \mathcal{S}$ and all $w \in \mathcal{W}_D$ it holds that $A_D x + E_D w \in \mathcal{S}$. A subset $\mathcal{S}$ of $\mathbb{R}^n$ is the minimal RPI set for uncertain linear dynamics* (2.1) *if and only if it is an RPI set and it is contained in all other RPI sets for uncertain linear dynamics* (2.1).

The effect of the additive, but bounded disturbances on linear dynamics (2.1) is best understood by considering related $k$–step reach sets $\mathcal{R}_D(\mathcal{X}, t_k)$ generated by the reach set map $\mathcal{R}_D(\cdot,\cdot)$, which, in view of (2.2), is defined for all subsets $\mathcal{X}$ of $\mathbb{R}^n$ and all $k \in \mathbb{Z}_{\geq 0},\ k > 0$, by:

$$\mathcal{R}_D(\mathcal{X}, t_k) := A_D^k \mathcal{X} \oplus \bigoplus_{i=0}^{k-1} A_D^i E_D \mathcal{W}_D \\ \text{with } \mathcal{R}_D(\mathcal{X}, t_0) := \mathcal{X}. \tag{2.3}$$

The reach set map $\mathcal{R}_D(\cdot,\cdot)$ is a semi–group. Thus, for any subset $\mathcal{X}$ of $\mathbb{R}^n$ and for all $i \in \mathbb{Z}_{\geq 0}$ and all $j \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{R}_D(\mathcal{X}, t_i + t_j) = \mathcal{R}_D(\mathcal{R}_D(\mathcal{X}, t_i), t_j). \tag{2.4}$$

The reach set map $\mathcal{R}_D(\cdot,\cdot)$ preserves both compactness and convexity. In fact, if $\mathcal{X}$ is either a $C$– or a proper $C$–set, the $k$–step reach sets $\mathcal{R}_D(\mathcal{X}, t_k)$ are guaranteed to be $C$–sets for all $k$ and proper $C$–sets for all large enough $k$. Furthermore, the reach set map $\mathcal{R}_D(\cdot,\cdot)$ is a monotone function in the first argument for all $t_k,\ k \in \mathbb{Z}_{\geq 0}$:

$$\mathcal{X} \subseteq \mathcal{Y} \Rightarrow \mathcal{R}_D(\mathcal{X}, t_k) \subseteq \mathcal{R}_D(\mathcal{Y}, t_k). \tag{2.5}$$

Thus, the reach set map preserves both positive invariance, i.e., for all $k \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{R}_D(\mathcal{X}, T) \subseteq \mathcal{X} \Rightarrow \mathcal{R}_D(\mathcal{X}, t_{k+1}) \subseteq \mathcal{R}_D(\mathcal{X}, t_k), \tag{2.6}$$

and positive anti–invariance, i.e., for all $k \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{X} \subseteq \mathcal{R}_D(\mathcal{X}, T) \Rightarrow \mathcal{R}_D(\mathcal{X}, t_k) \subseteq \mathcal{R}_D(\mathcal{X}, t_{k+1}). \tag{2.7}$$

Because of the semi–group property, the $k$–step reach sets can be characterized by iterating 1–step reach set map $\mathcal{R}_D(\cdot, T)$ given, for all subsets $\mathcal{X}$ of $\mathbb{R}^n$, by

$$\mathcal{R}_D(\mathcal{X}, T) := A_D \mathcal{X} \oplus E_D \mathcal{W}_D, \tag{2.8}$$

and by considering induced linear set–dynamics specified, for all $k \in \mathbb{Z}_{\geq 0}$, by:

$$\mathcal{X}(t_{k+1}) = \mathcal{R}_D(\mathcal{X}(t_k), T), \text{ or equivalently by} \\ \mathcal{X}(t_{k+1}) = A_D \mathcal{X}(t_k) \oplus E_D \mathcal{W}_D, \tag{2.9}$$

where, for any $k \in \mathbb{Z}_{\geq 0}$, $\mathcal{X}(t_k)$ is the $k$–step reach set from a given set $\mathcal{X} =: \mathcal{X}(0)$.

Indeed, positive invariance, stability and convergence properties of linear set–dynamics (2.9) reveal fundamental properties of the uncertain linear dynamics (2.1). For example, a subset $\mathcal{S}$ of $\mathbb{R}^n$ is an RPI set (see (2.2)) for uncertain linear dynamics (2.1) if and only if it is a positively invariant (PI) set for linear set–dynamics (2.9):

$$\mathcal{R}_D(\mathcal{S}, T) \subseteq \mathcal{S} \text{ i.e., } A_D \mathcal{S} \oplus E_D \mathcal{W}_D \subseteq \mathcal{S}. \tag{2.10}$$

More importantly, the fixed point set equation,

$$\mathcal{R}_D(\mathcal{S}, T) = \mathcal{S} \text{ i.e., } A_D \mathcal{S} \oplus E_D \mathcal{W}_D = \mathcal{S} \tag{2.11}$$

provides a necessary and sufficient condition for minimality of RPI sets for uncertain linear dynamics (2.1). The most important properties are summarized by:

**Theorem 1** *Suppose Assumption 1 holds. The unique solution to the fixed point set equation* (2.11) *is a proper $C$–set given explicitly by*

$$\mathcal{X}_{D\infty} = \bigoplus_{k=0}^{\infty} A_D^k E_d \mathcal{W}_D. \tag{2.12}$$

*The set $\mathcal{X}_{D\infty}$ is an exponentially stable attractor for linear set–dynamics* (2.9) *with the basin of attraction being the entire space of the compact subsets in $\mathbb{R}^n$.*

It is an important fact that the unique solution to the fixed point set equation (2.11), namely the set $\mathcal{X}_{D\infty}$ of (2.12), is the DT minimal RPI set, i.e., it is the minimal RPI set for uncertain linear dynamics (2.1). Attractivity of the set $\mathcal{X}_{D\infty}$ of (2.12) for linear set–dynamics (2.9) asserts that any state trajectory generated by uncertain linear dynamics (2.1) converges exponentially fast to the set $\mathcal{X}_{D\infty}$ of (2.12) and remains confined therein.

## 3 The CT forward reach and minimal RPI sets

Consider linear dynamics described, for all $t \in \mathbb{R}_{\geq 0}$, by:

$$\dot{x}(t) = A_C x(t) + E_C w(t) \text{ with} \\ w(t) \in \mathcal{W}_C, \tag{3.1}$$

where, for any $t \in \mathbb{R}_{\geq 0}$, $x(t) \in \mathbb{R}^n$ is the value of the state, $\dot{x}(t)$ is the value of the time derivative of the state, and $w(t) \in \mathbb{R}^p$ is the value of the disturbance at time $t$, while the matrices $A_C \in \mathbb{R}^{n \times n}$ and $E_C \in \mathbb{R}^{n \times p}$ and the set $\mathcal{W}_C \subset \mathbb{R}^p$ are known exactly.

**Assumption 2** *The matrix $A_C$ is strictly stable (i.e., $\sigma(A_C)$ lies in the interior of the left half of the complex plane). The matrix pair $(A_C, E_C)$ is controllable. The set $\mathcal{W}_C$ is a proper $C$–set in $\mathbb{R}^p$.*

For any $\delta \geq 0$, the admissible disturbance functions $w(\cdot)$ in (3.1) are Lebesgue measurable functions from time interval $[0, \delta]$ to the set $\mathcal{W}_C$. Consequently, for any initial state $x(0) = x$, any time $\delta \geq 0$ and any admissible disturbance function $w(\cdot) : [0, \delta] \to \mathcal{W}_C$ there is a unique state trajectory satisfying (3.1). In particular, the corresponding state trajectory satisfies, for all $t \in [0, \delta]$,

$$x(t) = e^{tA_C}x + \int_0^t e^{(t-\tau)A_C}E_C w(\tau)d\tau, \qquad (3.2)$$

where the integral in (3.2) is a standard point–valued (vector–valued) Lebesgue integral. Thus, in the CT setting, the notions of the RPI and minimal RPI sets are summarized via the following.

**Remark 2** *A subset $\mathcal{S}$ of $\mathbb{R}^n$ is an RPI set for uncertain linear dynamics* (3.1) *if and only if for all $x \in \mathcal{S}$, all $\delta \geq 0$ and all admissible disturbance function $w(\cdot) : [0, \delta] \to \mathcal{W}_C$, the related state trajectories, specified by* (3.2)*, satisfy, for all $t \in [0, \delta]$, $x(t) \in \mathcal{S}$. A subset $\mathcal{S}$ of $\mathbb{R}^n$ is the minimal RPI set for uncertain linear dynamics* (3.1) *if and only if it is an RPI set and it is contained in all other RPI sets for uncertain linear dynamics* (3.1)*.*

In this setting, the associated reach set map $\mathcal{R}_C(\cdot, \cdot)$ is specified, for any subset $\mathcal{X}$ of $\mathbb{R}^n$ and any time $t \geq 0$, by:

$$\mathcal{R}_C(\mathcal{X}, t) = e^{tA_C}\mathcal{X} \oplus \int_0^t e^{\tau A_C}E_C\mathcal{W}_C d\tau. \qquad (3.3)$$

The integral in (3.3) is the Aumann integral [12–14]. Formally, for any $t \geq 0$, the Aumann integral $\int_0^t e^{\tau A_C}E_C\mathcal{W}_C d\tau$ is the set of all integrals $\int_0^t e^{\tau A_C}E_C w(\tau)d\tau$ as $w(\cdot)$ varies across the set of the Lebesgue measurable functions from time interval $[0, t]$ to the set $\mathcal{W}_C$. Hence, the reach set from $\mathcal{X}$ at any time $t \geq 0$, $\mathcal{R}_C(\mathcal{X}, t)$, is the set of all states $x(t)$ given by (3.2) that are solutions of (3.1) as the initial conditions $x$ vary within $\mathcal{X}$ and disturbance functions $w(\cdot)$ vary within the class of admissible disturbance functions.

The CT reach set map $\mathcal{R}_C(\cdot, \cdot)$ also enjoys semi–group property. Namely, for any subset $\mathcal{X}$ of $\mathbb{R}^n$ and for all $\tau_1 \in \mathbb{R}_{\geq 0}$ and all $\tau_2 \in \mathbb{R}_{\geq 0}$, it holds that

$$\mathcal{R}_C(\mathcal{X}, \tau_1 + \tau_2) = \mathcal{R}_C(\mathcal{R}_C(\mathcal{X}, \tau_1), \tau_2). \qquad (3.4)$$

The reach set map $\mathcal{R}_C(\cdot, \cdot)$ is continuous in time (w.r.t. Hausdorff distance) and it preserves both compactness and convexity. As a matter of fact, in view of a fundamental result on convexity of set–valued integrals [12, Theorem 1.], convexity of the reach set map $\mathcal{R}_C(\cdot, \cdot)$ is guaranteed under mere convexity of $\mathcal{X}$ (i.e., convexity of $\mathcal{W}_C$ and/or $E_C\mathcal{W}_C$ is not required). If $\mathcal{X}$ is either a $C$–or a proper $C$–set, the reach sets $\mathcal{R}_C(\mathcal{X}, t)$ are guaranteed to be $C$–sets for all $t \in \mathbb{R}_{\geq 0}$ and proper $C$–sets for all large enough $t \in \mathbb{R}_{\geq 0}$. The reach set map $\mathcal{R}_C(\cdot, \cdot)$ is also a monotone function in the first argument for all $t \in \mathbb{R}_{\geq 0}$. In this sense, for all $t \in \mathbb{R}_{\geq 0}$,

$$\mathcal{X} \subseteq \mathcal{Y} \Rightarrow \mathcal{R}_C(\mathcal{X}, t) \subseteq \mathcal{R}_C(\mathcal{Y}, t). \qquad (3.5)$$

Consequently, the reach set map preserves both positive invariance, i.e., for all $\delta > 0$,

$$\begin{gathered}\forall \tau \in [0, \delta],\, \mathcal{R}_C(\mathcal{X}, \tau) \subseteq \mathcal{X} \Rightarrow \\ \forall t \geq 0,\ \forall \tau \in [0, \delta],\, \mathcal{R}_C(\mathcal{X}, t + \tau) \subseteq \mathcal{R}_C(\mathcal{X}, t), \end{gathered} \qquad (3.6)$$

and positive anti–invariance, i.e., for all $\delta > 0$,

$$\begin{gathered}\forall \tau \in [0, \delta],\, \mathcal{X} \subseteq \mathcal{R}_C(\mathcal{X}, \tau) \Rightarrow \\ \forall t \geq 0,\ \forall \tau \in [0, \delta],\, \mathcal{R}_C(\mathcal{X}, t) \subseteq \mathcal{R}_C(\mathcal{X}, t + \tau). \end{gathered} \qquad (3.7)$$

Because of the semi–group property (3.4), the reach sets $\mathcal{R}_C(\mathcal{X}, t)$ can be constructed via basic period reach set map $\mathcal{R}_C(\cdot, T)$ given, for all subsets $\mathcal{X}$ of $\mathbb{R}^n$, by

$$\mathcal{R}_C(\mathcal{X}, T) := e^{TA_c}\mathcal{X} \oplus \int_0^T e^{\tau A_C}E_C\mathcal{W}_C d\tau, \qquad (3.8)$$

and by using, for all $k \in \mathbb{Z}_{\geq 0}$, the semi–group relations

$$\forall t \in [0, T],\ \mathcal{X}(t_k + t) = \mathcal{R}_C(\mathcal{X}(t_k), t), \text{ or, equivalently,}$$
$$\mathcal{X}(t_k + t) = e^{tA_c}\mathcal{X}(t_k) \oplus \int_0^t e^{\tau A_C}E_C\mathcal{W}_C d\tau, \qquad (3.9)$$

where, for any $t \geq 0$, $\mathcal{X}(t)$ is the reach set at time $t$ from a given set $\mathcal{X} =: \mathcal{X}(0)$.

**Remark 3** *A minor rewriting of the relation* (3.9) *provides a direct link of the CT reach sets with both the DT and SD variants. More precisely, the relation* (3.9) *satisfies at the sampling instances, for all $k \in \mathbb{Z}_{\geq 0}$,*

$$\begin{aligned}&\mathcal{X}(t_{k+1}) = \mathcal{R}_C(\mathcal{X}(t_k), T), \ \textit{or equivalently} \\ &\mathcal{X}(t_{k+1}) = e^{TA_c}\mathcal{X}(t_k) \oplus \int_0^T e^{\tau A_C}E_C\mathcal{W}_C d\tau, \end{aligned} \qquad (3.10)$$

*and in the interior of the intervals $\mathcal{T}_k$, for all $t \in (0, T)$*

$$\begin{aligned}&\mathcal{X}(t_k + t) = \mathcal{R}_C(\mathcal{X}(t_k), t), \ \textit{or equivalently} \\ &\mathcal{X}(t_k + t) = e^{tA_c}\mathcal{X}(t_k) \oplus \int_0^t e^{\tau A_C}E_C\mathcal{W}_C d\tau. \end{aligned} \qquad (3.11)$$

As in the DT case, positive invariance, stability and convergence properties of the reach set operator (3.3) are directly related to fundamental properties of the uncertain linear dynamics (3.1). In this sense, a subset $\mathcal{S}$ of $\mathbb{R}^n$ is an RPI set for uncertain linear dynamics (3.1) (as described below relation (3.2)) if and only if it is PI under reach set operator (3.3):

$$\forall t \in \mathbb{R}_{\geq 0},\ \mathcal{R}_C(\mathcal{S}, t) \subseteq \mathcal{S} \text{ i.e.,}$$
$$\forall t \in \mathbb{R}_{\geq 0},\ e^{tA_C}\mathcal{S} \oplus \int_0^t e^{\tau A_C} E_C \mathcal{W}_C d\tau \subseteq \mathcal{S}. \quad (3.12)$$

Due to semi–group property (3.4) of the reach set operator $\mathcal{R}_C(\cdot,\cdot)$, the above positive invariance conditions need only be verified over an arbitrarily small, but positive measure, time interval $[0, \delta]$ (e.g., $[0, T]$). In the CT case, the fixed point functional set equation

$$\forall t \in \mathbb{R}_{\geq 0},\ \mathcal{R}_C(\mathcal{S}, t) = \mathcal{S} \text{ i.e.,}$$
$$\forall t \in \mathbb{R}_{\geq 0},\ e^{tA_C}\mathcal{S} \oplus \int_0^t e^{\tau A_C} E_C \mathcal{W}_C d\tau = \mathcal{S}. \quad (3.13)$$

provides a necessary and sufficient condition for minimality of RPI sets for uncertain linear dynamics (3.1). The most important properties are summarized by:

**Theorem 2** *Suppose Assumption 2 holds. The unique solution to the fixed point functional set equation* (3.13) *is a proper $C$–set given explicitly by*

$$\mathcal{X}_{C\infty} = \int_0^\infty e^{\tau A_C} E_C \mathcal{W}_C d\tau. \quad (3.14)$$

*The set $\mathcal{X}_{C\infty}$ is an exponentially stable attractor for set–dynamics whose trajectories* (3.9) *are generated by the reach set map $\mathcal{R}_C(\cdot,\cdot)$ of* (3.3) *with the basin of attraction being the entire space of the compact subsets in $\mathbb{R}^n$.*

The Aumann integral in (3.14) is the limit, w.r.t. Hausdorff distance as $t \to \infty$, of the Aumann integrals $\int_0^t e^{\tau A_C} E_C \mathcal{W}_C d\tau$.

Similarly to the DT setting, the unique solution to the fixed point functional set equation (3.13), namely the set $\mathcal{X}_{C\infty}$ of (3.14) is the CT minimal RPI set, i.e., it is the minimal RPI set for uncertain linear dynamics (3.1). Attractivity of the set $\mathcal{X}_{C\infty}$ of (3.14) asserts that any state trajectory generated by uncertain linear dynamics (3.1) converges exponentially fast to the set $\mathcal{X}_{C\infty}$ of (3.14) and remains confined therein.

In light of Remark 2 and the semi–group property of the CT reach set $\mathcal{R}_C(\cdot,\cdot)$, an intuitive connection between the forms of the DT and CT mRPI sets is as follows.

**Corollary 1** *Suppose Assumption 2 holds. Then*

$$\mathcal{X}_{C\infty} = \bigoplus_{k=0}^{\infty} \left( e^{kTA_C} \int_0^T e^{\tau A_C} E_C \mathcal{W}_C d\tau \right). \quad (3.15)$$

## 4 The SD reach and minimal RPI sets

### *4.1 Basic Setting*

Consider a linear system described, for all $t \in \mathbb{R}_{\geq 0}$, by:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t) \text{ with}$$
$$w(t) \in \mathcal{W}_S, \quad (4.1)$$

where, for any time $t \in \mathbb{R}_{\geq 0}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $w(t) \in \mathbb{R}^p$ denote, respectively, state, control and disturbance values, while $\dot{x}(t)$ denotes the value of the state derivative with respect to time, while the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $E \in \mathbb{R}^{n\times p}$ and the set $\mathcal{W}_S \subset \mathbb{R}^p$ are known exactly. The linear system (4.1) is controlled via SD linear state feedback so that

$$\forall k \in \mathbb{Z}_{\geq 0},\ \forall t \in \mathcal{T}_k,\ u(t) := Kx(t_k), \quad (4.2)$$

where $K \in \mathbb{R}^{m\times n}$ is a given control gain matrix. We note that the SD feedback at each time $t$ is not a function of the state at instant $t$, rather it is a function of the state at the last sampling instant $t_k$.

For any $k \in \mathbb{Z}_{\geq 0}$, within the setting of SD system and control, the admissible disturbance functions $w(\cdot)$ in (4.1) are, like the controls $u(\cdot)$, piecewise constant right continuous functions from time interval $[0, t_k]$ to the set $\mathcal{W}_S$ so that

$$\forall k \in \mathbb{Z}_{\geq 0},\ \forall t \in \mathcal{T}_k,\ w(t) := w(t_k) \in \mathcal{W}_S, \quad (4.3)$$

i.e., maps $w(\cdot)$ are constant in sampling intervals $\mathcal{T}_k$ and right continuous at sampling instants $t_k$ for all $k \in \mathbb{Z}_{\geq 0}$. Such a class of disturbances captures adequately the actuation errors as well as noise related errors in SD measurements in (4.2) and it also represents a reasonably rich model for other and more general types of uncertainty. This important class of disturbances allows for a natural and relatively simple analysis of forward reachability and robust positive invariance as shown next. Nevertheless, the larger class of Lebesgue measurable disturbances could be considered instead; see Section 4.6.

To define SD solutions, let, for any $t \in [0, T]$,

$$A_d(t) := e^{tA}, \quad B_d(t) := \left( \int_0^t e^{\tau A} d\tau \right) B \text{ and}$$
$$E_d(t) := \left( \int_0^t e^{\tau A} d\tau \right) E, \quad (4.4)$$

where the related integrals are the standard matrix–valued integrals, and let also, for any $t \in [0, T]$,

$$A_S(t) := A_d(t) + B_d(t)K \text{ and } E_S(t) := E_d(t) \text{ and}$$
$$A_D := A_S(T) \text{ and } E_D := E_S(T). \quad (4.5)$$

**Assumption 3** *The sampling period $T$ is such that the matrix pair $(A_d(T), B_d(T))$ is controllable. The control matrix $K$ is such that the matrix $A_D$ is strictly stable (i.e., $\rho(A_D) < 1$) and the matrix pair $(A_D, E_D)$ is controllable. The set $\mathcal{W}_S$ is a proper $C$–set in $\mathbb{R}^p$.*

In view of (4.1)–(4.3), the SD solutions coincide with the DT solutions (2.2) at the sampling instances $t_k$ so that, for all $k \in \mathbb{Z}_{\geq 0},\ k > 0$,

$$x(t_k) = A_D^k x + \sum_{i=0}^{k-1} A_D^{k-1-i} E_D w(t_i) \text{ with}$$
$$x(t_0) = x. \tag{4.6}$$

During sampling intervals $\mathcal{T}_k,\ k \in \mathbb{Z}_{\geq 0}$, the SD solutions satisfy, for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$, the property that

$$x(t_k + t) = A_S(t)x(t_k) + E_S(t)w(t_k). \tag{4.7}$$

### *4.2 The SD Reach Set Map*

In light of relations (4.6) and (4.7), the SD reach set map $\mathcal{R}_S(\cdot,\cdot)$ takes a form identical to the one associated with the discrete–time reach set $\mathcal{R}_D(\cdot,\cdot)$ at the sampling instances $t_k$, so that for all subsets $\mathcal{X}$ in $\mathbb{R}^n$ and all $k \in \mathbb{Z}_{\geq 0},\ k > 0$,

$$\mathcal{R}_S(\mathcal{X}, t_k) := A_D^k \mathcal{X} \oplus \bigoplus_{i=0}^{k-1} A_D^i E_D \mathcal{W}_S$$
$$\text{with } \mathcal{R}_S(\mathcal{X}, t_0) := \mathcal{X}. \tag{4.8}$$

During sampling intervals $\mathcal{T}_k,\ k \in \mathbb{Z}_{\geq 0}$, the SD reach set map $\mathcal{R}_S(\cdot,\cdot)$ satisfies for all subsets $\mathcal{X}$ in $\mathbb{R}^n$, all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$,

$$\mathcal{R}_S(\mathcal{X}, t_k + t) = A_S(t)\mathcal{R}_S(\mathcal{X}, t_k) \oplus E_S(t)\mathcal{W}_S. \tag{4.9}$$

Evidently, the reach set map $\mathcal{R}_S(\cdot,\cdot)$ in the SD setting exhibits a more complicated topological behaviour than its DT and CT analogues $\mathcal{R}_D(\cdot,\cdot)$ and $\mathcal{R}_C(\cdot,\cdot)$. The main ramification of this fact is reflected in a rather convoluted limiting behaviour.

The SD reach set map $\mathcal{R}_S(\cdot,\cdot)$ is continuous in time (w.r.t. Hausdorff distance) and it preserves both compactness and convexity. In addition, if $\mathcal{X}$ is either a $C$– or a proper $C$–set, the SD reach sets $\mathcal{R}_S(\mathcal{X}, t_k)$ at sampling instances $t_k$ are guaranteed to be $C$–sets for all $k$ and proper $C$–sets for all large enough $k$. In this case, the sampled data reach sets $\mathcal{R}_S(\mathcal{X}, t)$ in sampling intervals $\mathcal{T}_k,\ k \in \mathbb{Z}_{\geq 0}$ are only guaranteed to be $C$–sets. Without additional requirements, the sets $\mathcal{R}_S(\mathcal{X}, t)$ can not be *a–priori* guaranteed to be proper $C$–sets for all $t \in \mathcal{T}_k$ no matter how large $k$ is taken.

To illustrate phenomena arising when studying other relevant properties of the SD reach set $\mathcal{R}_S(\cdot,\cdot)$ we utilize the following example throughout this section.

**Example (Setting)** *We employ an instance of the SD system* (4.1) *for which*

$$A = 2\pi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ B = E = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathcal{W}_S = [-1, 1].$$

*The related exact discretization yields, for all $t \in \mathbb{R}_{\geq 0}$,*

$$A_d(t) = \begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix} \text{ and}$$
$$B_d(t) = E_d(t) = \frac{\sqrt{2}}{2\pi} \begin{pmatrix} -\cos\left(\frac{\pi(8t+1)}{4}\right) + 1 \\ \sin\left(\frac{\pi(8t+1)}{4}\right) - 1 \end{pmatrix}.$$

*The sampling period is $T = 0.25s$. The linear feedback $K$ is a deadbeat controller for $(A_d(T), B_d(T))$. In this setting, $A_D$ has all of its eigenvalues equal to* 0 *and $(A_D, E_D)$ is controllable. Furthermore, $A_D^k = 0$ for $k \in \mathbb{Z}_{\geq 0},\ k \geq 2$.*

In view of the SD nature of control feedback, the SD reach set map $\mathcal{R}_S(\cdot,\cdot)$ *fails, in general, to be a semi–group.* More precisely, the *DT semi–group* property given, for all subsets $\mathcal{X}$ of $\mathbb{R}^n$, all $i \in \mathbb{Z}_{\geq 0}$ and all $j \in \mathbb{Z}_{\geq 0}$, by

$$\mathcal{R}_S(\mathcal{X}, t_i + t_j) = \mathcal{R}_S(\mathcal{R}_S(\mathcal{X}, t_i), t_j), \tag{4.10}$$

**is guaranteed to hold**. However, the *CT semi–group* property specified analogously to (3.4), and required to be true for all subsets $\mathcal{X}$ of $\mathbb{R}^n$, and all $\tau_1 \in \mathbb{R}_{\geq 0}$ and all $\tau_2 \in \mathbb{R}_{\geq 0}$, **is not guaranteed to hold**. That is, without additional structure and further conditions, we might have

$$\mathcal{R}_S(\mathcal{X}, \tau_1 + \tau_2) \neq \mathcal{R}_S(\mathcal{R}_S(\mathcal{X}, \tau_1), \tau_2). \tag{4.11}$$

**Example (Semi–group Property)** *The first illustrative part of the example demonstrates the lack of generic semi–group property. In particular, Figure 1 depicts, in dark color, the forward reach sets $\mathcal{R}_S(\{0\}, t)$ for $t \in [0, T]$. For each $t \in [0, T]$, these sets are lower–dimensional and, in fact, simply rotated 1–D intervals in the underlying 2–D state space (and the origin at time* 0*). The figure also shows the forward reach sets $\mathcal{R}_S(\{0\}, t)$ and $\mathcal{R}_S(\mathcal{R}_S(\{0\}, T/2), t)$ for $t \in [0, T/2]$. The former sets $\mathcal{R}_S(\{0\}, t),\ t \in [0, T/2]$ are, as above, depicted in dark color, while the latter sets $\mathcal{R}_S(\mathcal{R}_S(\{0\}, T/2), t)$ for $t \in [0, T/2]$ are shown using transparent and lighter gray–scale shading. The latter sets are 2–D polytopes with* 4 *vertices for each time $t$ in $(0, T/2]$ and, thus, for all times $t \in (0, T/2]$, we have*

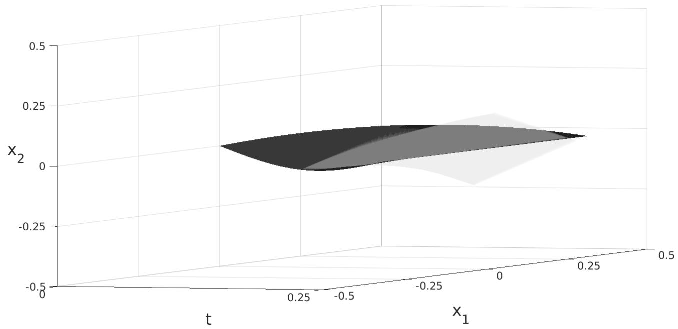

Fig. 1. Reach Sets $\mathcal{R}_S(\{0\}, t)$ and $\mathcal{R}_S(\mathcal{R}_S(\{0\}, T/2), t)$.

*that* $\mathcal{R}_S(\{0\}, T/2 + t) \neq \mathcal{R}_S(\mathcal{R}_S(\{0\}, T/2), t)$ *illustrating the asserted lack of generic semi–group property of the SD reach set map* $\mathcal{R}_S(\cdot, \cdot)$.

The SD reach set map $\mathcal{R}_S(\cdot, \cdot)$ remains monotone in the first argument for all $t \in \mathbb{R}_{\geq 0}$. In this sense, for all $t \in \mathbb{R}_{\geq 0}$,

$$\mathcal{X} \subseteq \mathcal{Y} \Rightarrow \mathcal{R}_S(\mathcal{X}, t) \subseteq \mathcal{R}_S(\mathcal{Y}, t). \tag{4.12}$$

However, in the absence of the generic semi–group property, and despite the monotonicity in the first argument, the SD reach set map $\mathcal{R}_S(\cdot, \cdot)$ *is not guaranteed to preserve either positive invariance or positive anti–invariance.* More precisely, the DT positive invariance related implication

$$\begin{aligned} &\mathcal{R}_S(\mathcal{X}, T) \subseteq \mathcal{X} \Rightarrow \\ &\forall k \in \mathbb{Z}_{\geq 0}, \mathcal{R}_S(\mathcal{X}, t_{k+1}) \subseteq \mathcal{R}_S(\mathcal{X}, t_k) \end{aligned} \tag{4.13}$$

**is guaranteed to hold**. However, the CT positive invariance related implication, for $\delta \in (0, T]$, **is not guaranteed to hold**. That is, generically,

$$\begin{aligned} &\forall \tau \in [0, \delta], \mathcal{R}_S(\mathcal{X}, \tau) \subseteq \mathcal{X} \not\Rightarrow \\ &\forall t \geq 0, \ \forall \tau \in [0, \delta], \mathcal{R}_S(\mathcal{X}, t + \tau) \subseteq \mathcal{R}_S(\mathcal{X}, t) \end{aligned} \tag{4.14}$$

Likewise, the DT positive anti–invariance related implication

$$\begin{aligned} &\mathcal{X} \subseteq \mathcal{R}_S(\mathcal{X}, T) \Rightarrow \\ &\forall k \in \mathbb{Z}_{\geq 0}, \mathcal{R}_S(\mathcal{X}, t_k) \subseteq \mathcal{R}_S(\mathcal{X}, t_{k+1}) \end{aligned} \tag{4.15}$$

**is guaranteed to hold**, while the CT positive anti–invariance related implication, for $\delta \in (0, T]$, **is not guaranteed to hold**. That is, generically,

$$\begin{aligned} &\forall \tau \in [0, \delta], \mathcal{X} \subseteq \mathcal{R}_S(\mathcal{X}, \tau) \not\Rightarrow \\ &\forall t \geq 0, \ \forall \tau \in [0, \delta], \mathcal{R}_S(\mathcal{X}, t) \subseteq \mathcal{R}_S(\mathcal{X}, t + \tau) \end{aligned} \tag{4.16}$$

**Example (Anti–invariance Property)** *This part of the example illustrates that the SD reach set map* $\mathcal{R}_S(\cdot, \cdot)$ *does not preserve invariance and anti–invariance properties. In particular, the set* $\{0\}$ *is contained in the forward*

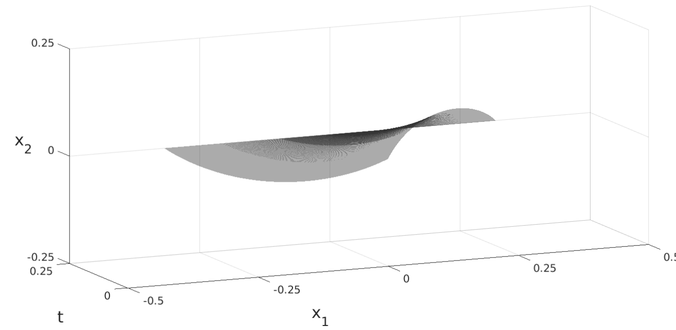

Fig. 2. Forward Reach Sets $\mathcal{R}_S(\{0\}, t)$.

*reach sets* $\mathcal{R}_S(\{0\}, t)$ *for all times* $t \in [0, T/2]$. *The forward reach sets* $\mathcal{R}_S(\{0\}, t)$, *for each* $t \in [0, T]$ *are all 1–D intervals and depicted using light gray–scale shading. As illustrated in Figure 2, these intervals, corresponding to the mentioned forward reach sets, have different lengths and are differently rotated for each time* $t \in [0, T]$. *Thus, for all times* $t$ *in the interval* $(0, T/2]$, *we have* $\mathcal{R}_S(\{0\}, t) \not\subseteq \mathcal{R}_S(\{0\}, T/2 + t)$ *demonstrating, in turn, that the anti–invariance property has not been preserved.*

In light of (4.8) and (4.9), the related SD reach sets satisfy, for all $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} &\forall t \in [0, T], \ \mathcal{X}(t_k + t) = \mathcal{R}_S(\mathcal{X}(t_k), t), \text{ or equivalently} \\ &\mathcal{X}(t_k + t) = A_S(t)\mathcal{X}(t_k) \oplus E_S(t)\mathcal{W}_S. \end{aligned} \tag{4.17}$$

**Remark 4** *A minor rearrangement of* (4.17) *reveals that the related SD reach sets satisfy at the sampling instances* $t_k$, *for all* $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} &\mathcal{X}(t_{k+1}) = \mathcal{R}_S(\mathcal{X}(t_k), T), \textit{ or equivalently} \\ &\mathcal{X}(t_{k+1}) = A_D\mathcal{X}(t_k) \oplus E_D\mathcal{W}_S, \end{aligned} \tag{4.18}$$

*and in the interior of the intervals* $\mathcal{T}_k$, *for all* $t \in (0, T)$,

$$\begin{aligned} &\mathcal{X}(t_k + t) = \mathcal{R}_S(\mathcal{X}(t_k), t), \textit{ or equivalently} \\ &\mathcal{X}(t_k + t) = A_S(t)\mathcal{X}(t_k) \oplus E_S(t)\mathcal{W}_S. \end{aligned} \tag{4.19}$$

### *4.3 The SD Robust Positive Invariance*

We have intentionally deferred discussing both RPI and minimal RPI sets in the SD setting. The main reason was to acquire necessary insights about the behaviour of the SD reach set map $\mathcal{R}_S(\cdot, \cdot)$ so that the adequate notions or RPI and minimal RPI sets can be introduced and discussed. In this sense, our previous analysis shows that the SD reach set map $\mathcal{R}_S(\cdot, \cdot)$ inherits (at sampling instances) properties of the DT reach set map $\mathcal{R}_D(\cdot, \cdot)$, while it might fail to inherit (in sampling intervals) properties of the CT reach set map $\mathcal{R}_C(\cdot, \cdot)$. In terms of SD RPI properties of the sets, the implication is that a demand for a subset $\mathcal{S}$ of $\mathbb{R}^n$ to be RPI in DT sense (i.e., RPI at the sampling instances):

$$\forall x \in \mathcal{S}, \forall w \in \mathcal{W}_S,\ A_D x + E_D w \in \mathcal{S}, \\ \text{i.e., } \mathcal{R}_S(\mathcal{S}, T) \subseteq \mathcal{S} \quad (4.20)$$

is natural and is, in fact, a minimal requirement to be imposed. However, the implication is also that a condition for a subset $\mathcal{S}$ of $\mathbb{R}^n$ to be RPI in CT sense (i.e., RPI at the sampling instances and in the sampling intervals):

$$\forall x \in \mathcal{S}, \forall w \in \mathcal{W}_S,\ \forall t \in [0,T],\ A_S(t)x + E_S(t)w \in \mathcal{S}, \\ \text{i.e., } \forall t \in [0,T],\ \mathcal{R}_S(\mathcal{S}, t) \subseteq \mathcal{S} \quad (4.21)$$

is *not natural and is, in fact, an overly conservative requirement.* Consequently, a natural notion of SD RPI should guarantee DT RPI and it should relax CT RPI but also facilitate it if it is attainable. Clearly, it is not possible to guarantee such a flexibility with utilization of a single set $\mathcal{S}$. Instead, similarly as it is done for set invariance under output feedback in [6], we introduce a generalized, and, in fact, relaxed, notion of RPI based on the utilization of a suitable family of sets.

**Definition 1** *A family of sets*

$$\mathfrak{S} := \{\mathcal{S}(t)\ :\ t \in [0,T]\}, \quad (4.22)$$

*where, for every $t \in [0,T]$, $\mathcal{S}(t)$ is a subset of $\mathbb{R}^n$, is an RPI family of sets for uncertain SD linear dynamics, specified via (4.1)–(4.3), if and only if*

$$\forall x \in \mathcal{S}(0),\ \forall w \in \mathcal{W}_S,\ \forall t \in [0,T], \\ A_S(t)x + E_S(t)w \in \mathcal{S}(t) \text{ and } \mathcal{S}(T) \subseteq \mathcal{S}(0). \quad (4.23)$$

Strictly speaking, the notion of RPI, as introduced in the above definition, is entirely compatible with the topological nature of the SD reach set map $\mathcal{R}_S(\cdot,\cdot)$, and it is, in fact, equivalent to weak PI of the collection of sets $\mathfrak{S}$ w.r.t. SD reach set map $\mathcal{R}_S(\cdot,\cdot)$:

$$\forall t \in [0,T],\ \mathcal{R}_S(\mathcal{S}(0), t) \subseteq \mathcal{S}(t) \text{ and} \\ \mathcal{S}(T) \subseteq \mathcal{S}(0). \quad (4.24)$$

**Remark 5** *Clearly, if there exists a subset $\mathcal{S}$ in $\mathbb{R}^n$ that verifies relations (4.20) and (4.21) (i.e., both DT and CT RPI) the related collection of sets $\mathfrak{S}$ satisfying (4.23) can be constructed by setting, for all $t \in [0,T]$, $\mathcal{S}(t) := \mathcal{S}$. Furthermore, a suitable family of sets $\mathfrak{S}$ satisfying (4.23) can be constructed easily given a subset $\mathcal{S}$ in $\mathbb{R}^n$ that verifies only relation (4.20) (i.e., only DT RPI). To this end, it suffices to put, for all $t \in [0,T]$,*

$$\mathcal{S}(t) := \mathcal{R}_S(\mathcal{S}, t), \text{ or equivalently,} \\ \mathcal{S}(t) := A_S(t)\mathcal{S} \oplus E_S(t)\mathcal{W}_S. \quad (4.25)$$

*We note that such a family of sets is as easy to detect and work with as usual DT RPI sets, namely its members $\mathcal{S}(t),\ t \in [0,T]$ (and hence family itself) are implicitly characterized by sets $\mathcal{S}$ and $\mathcal{W}_S$ as specified in (4.25).*

### *4.4 Minimality of SD RPI Sets*

We focus now on the corresponding limiting behaviour and an adequate notion of the minimality of RPI sets in SD setting. By Theorem 1, the set

$$\mathcal{X}_{S\infty} := \bigoplus_{k=0}^{\infty} A_D^k E_D \mathcal{W}_S \quad (4.26)$$

is a proper $C$–set in $\mathbb{R}^n$ and the unique solution to the fixed point set equation (2.11) for SD setting, i.e., $\mathcal{R}_S(\mathcal{S}, T) = \mathcal{S}$ or equivalently $A_D\mathcal{S} \oplus E_D\mathcal{W}_S = \mathcal{S}$. Furthermore, for any compact subset $\mathcal{S}$ in $\mathbb{R}^n$, the related sequence of the SD reach sets $\mathcal{R}_S(\mathcal{S}, t_k)$ at sampling instances $t_k,\ k \geq 0$ converges to $\mathcal{X}_{S\infty}$ exponentially fast w.r.t. Hausdorff distance. In fact, the set $\mathcal{X}_{S\infty}$ is the unique set that satisfies, for all $k \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{R}_S(\mathcal{X}_{S\infty}, t_{k+1}) = A_D\mathcal{R}_S(\mathcal{X}_{S\infty}, t_k) \oplus E_D\mathcal{W}_S \\ = \mathcal{R}_S(\mathcal{X}_{S\infty}, t_k) = \mathcal{X}_{S\infty}. \quad (4.27)$$

The compactness of $\mathcal{X}_{S\infty}$ and the continuity of the reach set $\mathcal{R}_S(\cdot,\cdot)$ in time w.r.t. Hausdorff distance guarantee that the SD reach set $\mathcal{R}_S(\cdot,\cdot)$ remains bounded and, hence, it preserves compactness in its limiting behavior. In particular, during sampling intervals $\mathcal{T}_k$, we have, for all $t \in [0,T)$,

$$\mathcal{R}_S(\mathcal{X}_{S\infty}, t_k + t) = A_S(t)\mathcal{R}_S(\mathcal{X}_{S\infty}, t_k) \oplus E_S(t)\mathcal{W}_S \\ = A_S(t)\mathcal{X}_{S\infty} \oplus E_S(t)\mathcal{W}_S. \quad (4.28)$$

Consequently, for any fixed $t \in [0,T)$, and any compact subset $\mathcal{S}$ of $\mathbb{R}^n$, the SD reach set $\mathcal{R}_S(\mathcal{S}, t_k + t)$ converges to $A_S(t)\mathcal{X}_{S\infty} \oplus E_S(t)\mathcal{W}_S$ exponentially fast w.r.t. the Hausdorff distance (as $k$ and, hence, $t_k$ go to infinity).

**Example (Attractivity Property)** *This part of the example illustrates the above discussed attractivity properties. The forward reach sets $\mathcal{R}_S(\{0\}, t),\ t \in [0, 5T]$ (the*

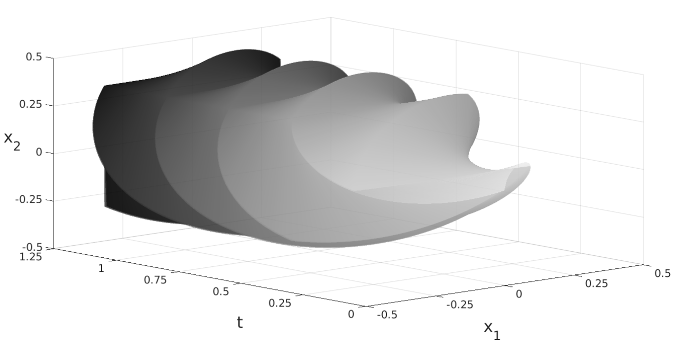


Fig. 3. Forward Reach Sets $\mathcal{R}_S(\{0\}, t)$.

*corresponding forward reach tube over the time interval*

$[0, 5T]$), *are plotted in Figure 3. using different levels of gray–scale shading (the darker color indicates the larger time t). In this case, the convergence occurs in 2 sampling periods since* $A_D^2 = 0$. *As evident by inspection of the figure, the forward reach sets* $\mathcal{R}_S(\{0\}, t)$, $t \in [2T, 5T]$ *exhibit periodic limiting behavior, as expected in light of the above discussion and relations* (4.25)–(4.27).

In view of above analysis, let, for all $t \in [0, T]$,

$$\mathcal{X}_{S\infty}(t) := A_S(t)\mathcal{X}_{S\infty} \oplus E_S(t)\mathcal{W}_S, \tag{4.29}$$

and define a compact collection of $C$–sets in $\mathbb{R}^n$

$$\mathfrak{X}_{S\infty} := \{\mathcal{X}_{S\infty}(t) \ : \ t \in [0, T]\}. \tag{4.30}$$

The following result follows immediately from Thoerem 1 and above constructions, and it sets a basis for a suitably notion of the minimality of RPI sets in SD case.

**Proposition 1** *Suppose Assumption 3 holds. Let* $\mathfrak{S} = \{\mathcal{S}(t) \ : \ t \in [0, T]\}$ *be any RPI family of sets for uncertain SD linear dynamics. Consider also the family of sets* $\mathfrak{X}_{S\infty} = \{\mathcal{X}_{S\infty}(t) \ : \ t \in [0, T]\}$ *given by* (4.30). *Then,* $\mathfrak{X}_{S\infty}$ *is an RPI family of sets for uncertain SD linear dynamics and, furthermore,*

$$\forall t \in [0, T], \ \mathcal{X}_{S\infty}(t) \subseteq \mathcal{S}(t). \tag{4.31}$$

The above proposition justifies the following natural, “pointwise–in–time–over–the–sampling–interval”, notion of minimal family of RPI sets within the setting of uncertain SD dynamics.

**Definition 2** *A family of sets*

$$\mathfrak{S}_\infty := \{\mathcal{S}_\infty(t) \ : \ t \in [0, T]\}, \tag{4.32}$$

*where, for every* $t \in [0, T]$, $\mathcal{S}_\infty(t)$ *is a subset of* $\mathbb{R}^n$, *is the minimal RPI family of sets for uncertain SD linear dynamics if and only if* $\mathfrak{S}_\infty$ *is an RPI family of sets for uncertain SD linear dynamics and, for any RPI family* $\mathfrak{S} = \{\mathcal{S}(t) \ : \ t \in [0, T]\}$ *of sets for uncertain SD linear dynamics, it holds that*

$$\forall t \in [0, T], \ \mathcal{S}_\infty(t) \subseteq \mathcal{S}(t). \tag{4.33}$$

**Example (Minimality Property)** *This part of the example illustrates the minimal RPI family* $\mathfrak{X}_{S\infty}$ *and its related invariance properties. The forward reach sets* $\mathcal{R}_S(\mathcal{X}_{S_\infty}, t)$, $t \in [0, 5T]$ *are, as in the previous part of the example, plotted in Figure 4. using different levels of gray–scale shading. The forward reach sets* $\mathcal{R}_S(\mathcal{X}_{S_\infty}, t)$, $t \in [0, 5T]$ *exhibit periodic behavior and never leave the minimal RPI family* $\mathfrak{X}_{S\infty}$. *This behavior is expected in view of our analysis.*

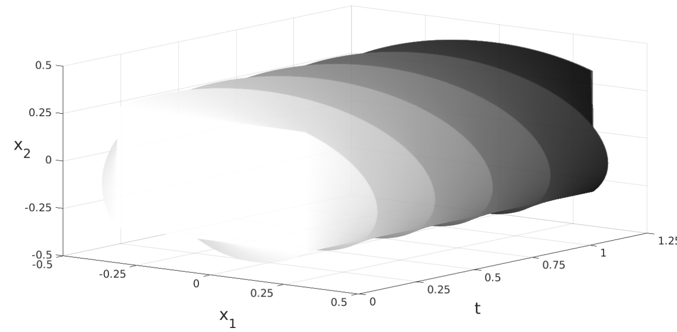


Fig. 4. Minimal RPI Family $\mathfrak{X}_{S\infty}$.

Theorem 1, Proposition 1 and the analysis preceding it verify a conclusion that the minimal RPI family of sets for uncertain SD linear dynamics is unique and well–defined. In view of (4.25), the minimal family of RPI sets for uncertain SD linear dynamics is entirely determined by the set $\mathcal{S}$ defined by fixed point set equation

$$\mathcal{R}_S(\mathcal{S}, T) = \mathcal{S} \text{ i.e., } A_D\mathcal{S} \oplus E_D\mathcal{W}_S = \mathcal{S}. \tag{4.34}$$

Indeed, once the unique solution to (4.34) is identified, it suffices to use (4.25) and put, for all $t \in [0, T]$, $S(t) = \mathcal{R}_S(\mathcal{S}, t) = A_S(t)\mathcal{S} \oplus E_S\mathcal{W}_S$. This construction verifies the uniqueness and minimality of the family of sets $\mathfrak{X}_{S\infty}$ specified by (4.30) as summarized by:

**Theorem 3** *Suppose Assumption 3 holds. Then:*

(*i*) *The unique solution to the fixed point set equation* (4.34) *is a proper* $C$*–set given explicitly by* (4.26).
(*ii*) *The family of sets* $\mathfrak{X}_{S\infty}$ *specified by* (4.30) *is an exponentially stable weak upper–attractor*[2] *for set–dynamics whose trajectories* (4.17) *are generated by the reach set map* $\mathcal{R}_S(\cdot, \cdot)$ *of* (4.8) *and* (4.9) *with the basin of attraction being the entire space of the compact subsets in* $\mathbb{R}^n$.
(*iii*) *The family of sets* $\mathfrak{X}_{S\infty}$ *is the minimal RPI family of sets, as specified in Definition 2, for uncertain SD linear dynamics given via* (4.1)–(4.3).

**Remark 6** *We close this section by pointing out that, under monotonicity of the reach set map* $\mathcal{R}_S(\cdot, \cdot)$ *in the second argument, i.e., for all subsets* $\mathcal{S}$ *in* $\mathbb{R}^n$, *all* $\tau_1 \in$

[2] The notion of an exponentially stable upper–attractor means that the forward reach sets $\mathcal{R}_S(\mathcal{X}, t)$ exhibit stable behavior and upper–converge w.r.t. Hausdorff distance to the family of sets $\mathfrak{X}_{S\infty}$ as $t \to \infty$ for all compact subsets $\mathcal{X}$ of $\mathbb{R}^n$, i.e., the function $d(\mathcal{X}, t) := \min_{\mathcal{Y}}\{\mathrm{H}_{\mathcal{L}}(\mathcal{Y}, \mathcal{R}_S(\mathcal{X}, t)) \ : \ \mathcal{Y} \in \mathfrak{X}_{S\infty}\}$, defined for all compact subsets $\mathcal{X}$ of $\mathbb{R}^n$ and all times $t \in \mathbb{R}_{\geq 0}$, vanishes as $t \to \infty$ for all compact subsets $\mathcal{X}$ of $\mathbb{R}^n$. The notion is weak since the SD reach set is not a semi–group so that only $\mathcal{R}_S(\mathcal{X}_{S\infty}(0), t) \in \mathfrak{X}_{S\infty}$ for all $t \geq 0$ is guaranteed in light of the generalized characterization of the family attractor as specified via (4.26)–(4.30).

$\mathbb{R}_{\geq 0}$ *and all* $\tau_2 \in \mathbb{R}_{\geq 0}$,

$$\tau_1 \leq \tau_2 \Rightarrow \mathcal{R}_S(\mathcal{S}, \tau_1) \subseteq \mathcal{R}_S(\mathcal{S}, \tau_2), \tag{4.35}$$

*the sets* $\mathcal{X}_{S\infty}(t),\ t \in [0, T]$ *are identical and equal to the set* $\mathcal{X}_{S\infty}$ *of* (4.26) *so that the family of sets* $\mathfrak{X}_{S\infty}$ *can be reduced to a singleton set* $\mathcal{X}_{S\infty}$ *that also becomes a strong attractor (instead of a weak upper–attractor) for the related set–dynamics of the SD reach sets.*

### *4.5 Simpler, Approximate and Guaranteed, Notions*

A simpler, approximate and guaranteed, robust positive invariance notions can be obtained by combing the safety over the sampling period and robust positive invariance at the sampling instances. More specifically, it is possible to employ a pair of sets $(\mathcal{I}, \mathcal{O})$ and invoke the following set of conditions:

$$\forall t \in [0, T],\ \mathcal{R}_S(\mathcal{I}, t) \subseteq \mathcal{O} \text{ and } \mathcal{R}_S(\mathcal{I}, T) \subseteq \mathcal{I}. \tag{4.36}$$

This construction is captured by the following "uniform–over–the–sampling–interval" notion.

**Definition 3** *A pair of sets* $(\mathcal{I}, \mathcal{O})$*, where* $\mathcal{I}$ *and* $\mathcal{O}$ *are subsets of* $\mathbb{R}^n$*, is a safe RPI pair of sets for uncertain SD linear dynamics, specified via* (4.1)–(4.3)*, if and only if*

$$\begin{aligned} &\forall x \in \mathcal{I},\ \forall w \in \mathcal{W}_S,\ \forall t \in [0, T], \\ &A_S(t)x + E_S(t)w \in \mathcal{O} \text{ and } A_D x + E_D w \in \mathcal{I}. \end{aligned} \tag{4.37}$$

Clearly, the "uniform–over–the–sampling–interval" RPI notion is less flexible than "pointwise–in–time–over–the–sampling–interval" RPI notion. It is worth pointing out that it is possible to transition from one notion to the other. In particular, given a family of RPI sets $\mathfrak{S} := \{\mathcal{S} : t \in [0, T]\}$ satisfying Definition 1, a corresponding safe RPI pair of sets $(\mathcal{I}, \mathcal{O})$ satisfying Definition 3 can be obtained by simply setting:

$$\mathcal{I} := \mathcal{S}(0) \text{ and } \mathcal{O} := \bigcup_{t \in [0,T]} \mathcal{S}(t). \tag{4.38}$$

Note that any set $\mathcal{O}$ such that $\bigcup_{t \in [0,T]} \mathcal{S}(t) \subseteq \mathcal{O}$ can be also employed. This transition is generally less "loose" than the transition from a safe RPI pair of sets $(\mathcal{I}, \mathcal{O})$ satisfying Definition 3 to a family of RPI sets $\mathfrak{S} := \{\mathcal{S} : t \in [0, T]\}$ satisfying Definition 1 via

$$\mathcal{S}(0) = \mathcal{S}(T) := \mathcal{I} \text{ and } \forall t \in (0, T),\ \mathcal{S}(t) := \mathcal{O}. \tag{4.39}$$

The "uniform–over–the–sampling–interval" notion of minimal safe RPI pairs of sets within the setting of uncertain SD dynamics is as follows.

**Definition 4** *A pair of sets* $(\mathcal{I}_\infty, \mathcal{O}_\infty)$*, where* $\mathcal{I}_\infty$ *and* $\mathcal{O}_\infty$ *are subsets of* $\mathbb{R}^n$*, is the minimal safe RPI pair of sets for uncertain SD linear dynamics if and only if* $(\mathcal{I}_\infty, \mathcal{O}_\infty)$ *is a safe RPI pair of sets for uncertain SD linear dynamics and, for any other safe RPI pair of sets* $(\mathcal{I}, \mathcal{O})$ *for uncertain SD linear dynamics, it holds that:*

$$\mathcal{I}_\infty \subseteq \mathcal{I} \text{ and } \mathcal{O}_\infty \subseteq \mathcal{O}. \tag{4.40}$$

The existence and uniqueness of the minimal safe RPI pair of sets follows as an immediate consequence of Proposition 1, Theorem 3 and Definition 4.

**Corollary 2** *Suppose Assumption 3 holds and let*

$$\begin{aligned} &\mathcal{I}_\infty := \mathcal{X}_{S\infty}(0) = \mathcal{X}_{S\infty} \text{ and} \\ &\mathcal{O}_\infty := \bigcup_{t \in [0,T]} \mathcal{X}_{S\infty}(t) = \bigcup_{t \in [0,T]} \left(A_S(t)\mathcal{X}_{S\infty} \oplus E_S(t)\mathcal{W}_S\right). \end{aligned}$$

*The pair of sets* $(\mathcal{I}_\infty, \mathcal{O}_\infty)$ *is the unique minimal safe RPI pair of sets* $(\mathcal{I}_\infty, \mathcal{O}_\infty)$ *for uncertain SD linear dynamics.*

**Example (Minimality of the safe RPI pairs)** *The DT minimal RPI set* $\mathcal{X}_{S_\infty} = A_D\mathcal{W}_S \oplus \mathcal{W}_S$ *is a polytope with 4 vertices and it is the corresponding set* $\mathcal{I}_\infty$ *(plotted in white in Figure 5.), while the set* $\mathcal{O}_\infty$ *is simply the Euclidean norm ball whose radius is equal to the norm of the vertices of the DT minimal RPI set* $\mathcal{X}_{S_\infty}$*. These sets are obtained by using Corollary 2. As expected in light of our analysis, the forward reach sets* $\mathcal{R}_S(\mathcal{X}_{S_\infty}, t),\ t \in [0, 5T]$ *starting from* $\mathcal{I}_\infty = \mathcal{X}_{S_\infty}$ *remain within the minimal RPI family* $\mathfrak{X}_{S\infty}$ *are contained in the set* $\mathcal{O}_\infty$ *for all times as illustrated in the figure. Namely, the forward reach sets* $\mathcal{R}_S(\mathcal{X}_{S_\infty}, t),\ t \in [0, 5T]$ *are contained in the cylinder* $\mathcal{O}_\infty \times [0, 5T]$ *(shown using a very light gray–scale shading); The related inclusion is satisfied at each corresponding time instance.*

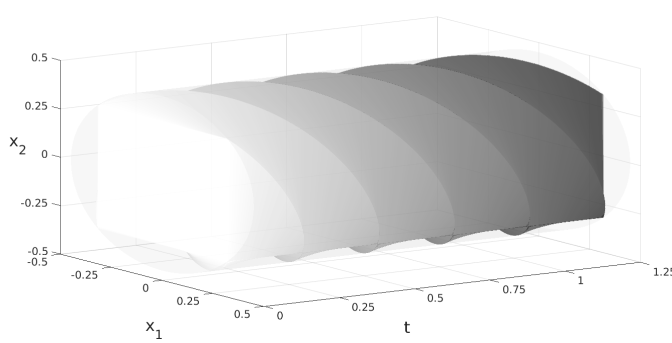


Fig. 5. Minimal Safe RPI Pair $(\mathcal{I}_\infty, \mathcal{O}_\infty)$.

### *4.6 Lebesgue Measurable Disturbances for SD Systems*

The introduced notions apply to the case of Lebesgue measurable disturbances acting upon the SD system controlled with sampled data control feedback. The developed frameworks, properties and notions apply directly with relatively minor changes necessary to account for allowing a richer class of disturbance to affect the SD systems. In particular, the solutions have to be integrated with such an assumption in mind and the remaining analysis needs to be modified. Thus, the time–varying SD disturbance sets $E_s(t)\mathcal{W}_S$ should be replaced with $\mathcal{W}_L(t) := \int_0^t e^{\tau A} E\mathcal{W}_S d\tau$ for all $t \in [0, T]$ (and throughout all time intervals $\mathcal{T}_k,\ k \in \mathbb{Z}_{\geq 0}$). The latter time–varying disturbance sets $\mathcal{W}_L(t),\ t \in [0, T]$ are obtained by Aumann integration and they account exactly for the presence of Lebesgue measurable disturbances. The remaining changes are relatively direct and dominantly notational ones and thus, are not elaborated on in more details in this article.

## 5 Closing Discussion

We have revisited forward reachability and robust positive invariance analyses for DT and CT problems in order to develop novel techniques for studying forward reachability and robust positive invariance of SD systems. We summarized key exisiting results regarding forward reach, RPI and minimal RPI sets for the DT and CT cases and developed new results for the SD case that revealed substantial structural differences to the DT and CT cases. In particular, we introduced topologically compatible notions for the SD forward reach, RPI and minimal RPI sets and we addressed and enhanced computational aspects associated with these notions by complementing them with approximate, but guaranteed, and numerically more plausible notions.

The results developed here in the SD setting are relevant in constrained control schemes that use CT plant models, which in fact overall are SD systems, including (sampled-data) model predictive control [15,16] and reference governors [17]. Additionally, the reported results also provide, currently unavailable, mathematical foundations for developing topologically appropriate frameworks for the analysis and computation of the backward reach and the maximal RPI sets for SD systems. These important research questions are currently under investigation and the findings will be reported elsewhere.


## Acknowledgments

This research started while Saša V. Raković was a visiting scholar with the Center for Space Research of the University of Texas at Austin, USA. The research was carried out while F. A. C. C. Fontes was visiting CESG, Department of Electrical & Computer Engineering, Texas A&M University, College Station, USA.


## References


[1] F. C. Schweppe. *Uncertain Dynamic Systems.* Prentice Hall, Englewood Cliffs, NJ, 1973.

[2] F. L. Chernousko. *State Estimation of Dynamic systems.* CRC Press, Boca Raton, 1994.

[3] A.B. Kurzhanski and I. Vályi. *Ellipsoidal Calculus for Estimation and Control.* Systems & Control: Foundations & Applications. Birkhauser, Boston, Basel, Berlin, 1997.

[4] F. Blanchini and S. Miani. *Set–Theoretic Methods in Control.* Systems & Control: Foundations & Applications. Birkhauser, Boston, Basel, Berlin, 2008.

[5] J. P. Aubin. *Viability theory.* Springer Science & Business Media, 2009.

[6] Z. Artstein and S. V. Raković. Set Invariance Under Output Feedback : A Set–Dynamics Approach. *International Journal of Systems Science*, 42(4):539–555, 2011.

[7] A.B. Kurzhanski and P. Varaiya. *Dynamics and Control of Trajcetory Tubes: Theory and Computation.* Systems & Control: Foundations & Applications. Birkhauser, Springer, Cham, Heidelberg, New York, Dordrecht, London, 2014.

[8] I. V. Kolmanovsky and E. G. Gilbert. Theory and Computation of Disturbance Invariant Sets for Discrete–Time Linear Systems. *Mathematical Problems in Egineering*, 4:317–367, 1998.

[9] S. V. Raković. Minkowski Algebra and Banach Contraction Principle in Set Invariance for Linear Discrete Time Systems. In *Proceedings of 46th IEEE Conference on Decision and Control, CDC 2007*, New Orleans, LA, USA, December 2007.

[10] Z. Artstein and S. V. Raković. Feedback and Invariance under Uncertainty via Set Iterates. *Automatica*, 44(2):520–525, 2008.

[11] S. V. Raković and K. I. Kouramas. Invariant Approximations of the Minimal Robust Positively Invariant Set via Finite Time Aumann Integrals. In *Proceedings of 46th IEEE Conference on Decision and Control, CDC 2007*, New Orleans, LA, USA, December 2007.

[12] R. J. Aumman. Integrals of set–valued functions. *Journal of Mathematical Analysis and Applications*, 12(1):1–12, 1965.

[13] Z. Artstein. On the calculus of closed set–valued functions. *Indiana University Mathematics Journal*, 24(5):433–441, 1974.

[14] Z. Artstein. A calculus for set–valued maps and set–valued evolution equations. *Set–valued Analysis*, 3:213–261, 1995.

[15] F. A. C. C. Fontes. A general framework to design stabilizing nonlinear model predictive controllers. *Systems & Control Letters*, 42(2):127–143, 2001.

[16] F. A. C. C. Fontes, L. Magni, and É. Gyurkovics. Sampled–data model predictive control for nonlinear time–varying systems: Stability and robustness. In *Assessment and Future Directions of Nonlinear Model Predictive Control*, pages 115–129. Springer Berlin Heidelberg, 2007.

[17] I. V. Kolmanovsky, E. Garone, and S. Di Cairano. Reference and command governors: A tutorial on their theory and automotive applications. In *American Control Conference (ACC), 2014*, pages 226–241. IEEE, 2014.